THE KING'S SCHOOL

2007
Higher School Certificate
Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Answer each question in a separate booklet

Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

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Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{3 + \cos x}$ **3**

(b) (i) Evaluate $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \frac{x}{\sqrt{1 - x^2}}\, dx$ **3**

(ii) Evaluate $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \cos^{-1} x \;\, dx$ **2**

(c) (i) Find A if $\dfrac{2x + A}{x^2 + 1} - \dfrac{2}{x - 2} \equiv \dfrac{x - 12}{\left(x^2 + 1\right)\left(x - 2\right)}$ **1**

(ii) Evaluate $\displaystyle\int_0^1 \frac{x - 12}{\left(x^2 + 1\right)\left(x - 2\right)}\, dx$ **3**

(d) Find $\displaystyle\int \tan^3 x \;\, dx$ **3**

End of Question 1

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $z = a + i$ and $w = 1 + ai$, a real. Find

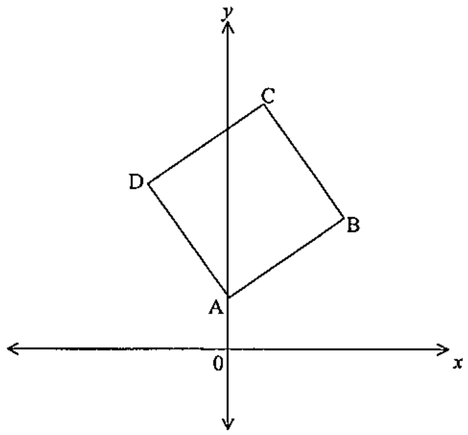
(i) $\left| \dfrac{z}{w} \right|$ **1**

(ii) $\arg zw$ **2**

(b) The point $P(x, y)$ represents the complex number z in the Argand diagram.

Sketch the locus of P if $Im\,(1 - i)\,z \geq 1$ **3**

(c)



The diagram shows the square ABCD in the complex plane.

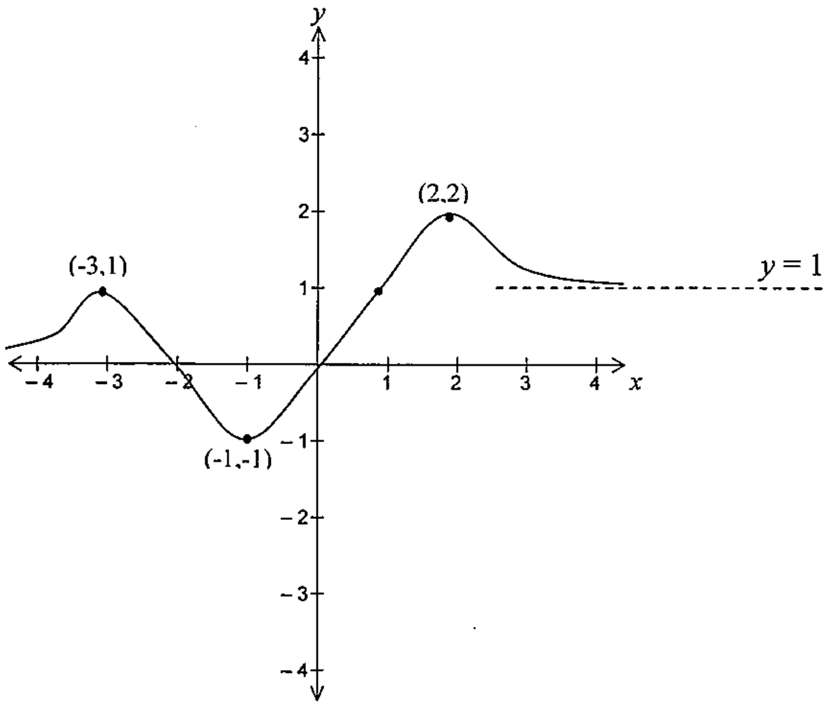
A represents the complex number i and B represents the complex number z.

(i) Find the complex number $\overrightarrow{AD}$ **2**

(ii) Hence or otherwise find the complex number represented by the point C. **1**

Question 2 continues next page

(d) The diagram shows the graph of $y = f(x)$



The lines $y = 1$ and the x axis are asymptotes.

Draw separate sketches of the graphs of:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = \ln f(x)$ **2**

(iii) $y = f(x + |x|)$ **2**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) $S(c, o)$ and $S^1(-c, o)$, where $c > 0$, are the foci of the hyperbola $x^2 - y^2 = 2$

Find S and sketch the hyperbola showing its foci, directrices, asymptotes and any intercepts made with the coordinate axes. **3**

(ii) $P(\sqrt{2}\sec\theta, \sqrt{2}\tan\theta)$ is a point in the first quadrant on the hyperbola $x^2 - y^2 = 2$ in (i).

A circle with centre P and radius PS is drawn.

(α) Find the length of the radius in simplest form. **2**

(β) The line S^1P cuts the circle at Q and R where Q is between S^1 and P.

It can be shown that $Q = \left(\dfrac{2}{\sqrt{2}\sec\theta + 1}, \dfrac{2\sqrt{2}\tan\theta}{\sqrt{2}\sec\theta + 1} \right)$

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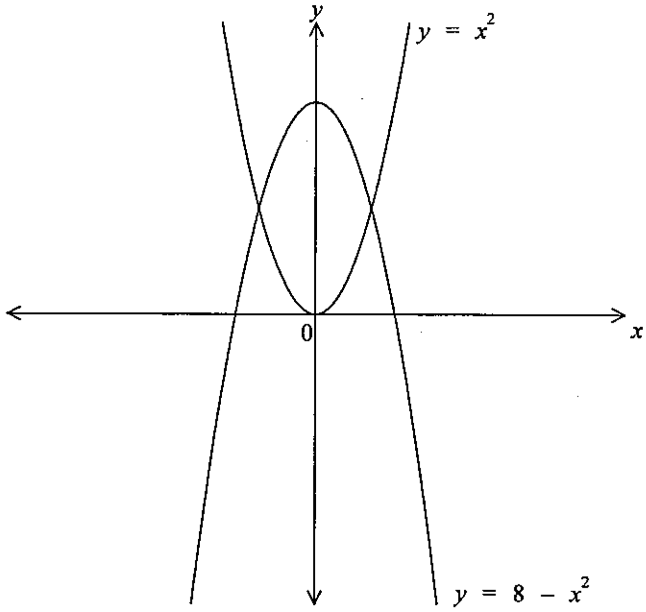
Prove that QS is parallel to the normal to the hyperbola at P. **3**

(γ) Explain why RS is parallel to the tangent to the hyperbola at P. **2**

Question 3 continues next page

(b)



The diagram shows the two parabolae $y = x^2$ and $y = 8 - x^2$

A solid is formed using the region enclosed between the two parabolae as its base.

Cross-sections parallel to the y axis and perpendicular to the xy plane are semi-circles where the diameters are in the base of the solid.

Prove that the volume of this solid is $\dfrac{256\pi}{15}$ cubic units. **5**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A particular curve passes through the origin and its derivative is given by $\frac{dy}{dx} = \sqrt{4y^2 + 1}$

(i) Prove that $\frac{d^2y}{dx^2} = 4y$ **2**

(ii) Use the table of standard integrals to find x as a function of y. **2**

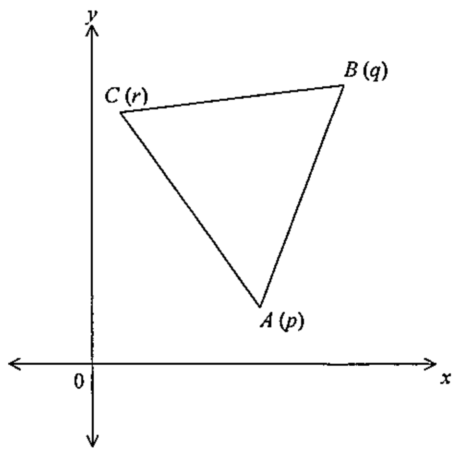
(b) (i) Sketch the region where $0 \le y \le x - x^2$ **1**

(ii) The region in (i) is revolved about the line $x = -1$

Use the method of cylindrical shells to find the volume of the solid of revolution generated. **4**

(c) (i) Express $\frac{1}{2}(1 + i\sqrt{3})$ in mod-arg form. **1**

(ii)



In the Argand diagram the points A, B, C represent the complex numbers p, q, r, respectively, where $r - p = \frac{1}{2}(1 + i\sqrt{3})(q - p)$

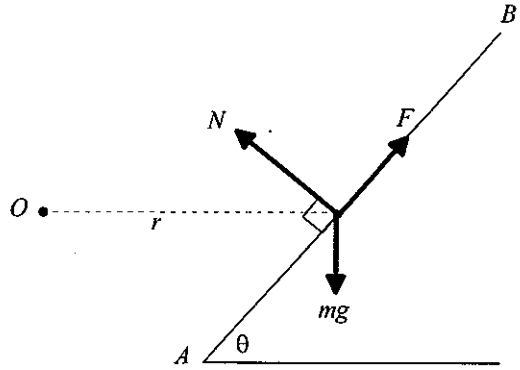
(α) Prove that $p - q = \frac{1}{2}(1 + i\sqrt{3})(r - q)$ **3**

(β) Deduce that $p^2 + q^2 + r^2 = pq + qr + rp$ **2**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The diagram shows the forces exerted on a car of mass m travelling at speed v on a banked circular track AB of radius r. The track is banked inwards at θ to the horizontal. The road exerts the normal force N at right angles to the road and there is a frictional force F exerted up the track.

(i) By resolving the forces in the direction BA, or otherwise, show that **2**

$$F = mg\sin\theta - \frac{mv^2}{r}\cos\theta$$

(ii) Deduce that $v^2 < gr\tan\theta$ **2**

(iii) Draw a diagram showing the forces on the car if $v^2 > gr\tan\theta$ **1**

(iv) Find an expression for N not involving F. **2**

(b) u, v, w are the roots of $x^3 + Ax + B = 0$

(i) Show that $u^2 + v^2 + w^2 = -2A$ **2**

(ii) Let $y = \frac{v}{w} + \frac{w}{v}$

Prove that $u^3 + 2Au - By = 0$ **2**

(iii) By using another equation involving u^3 show that $u = \frac{B}{A}(y + 1)$ **2**

(iv) Show that the equation with roots $\frac{v}{w} + \frac{w}{v}, \frac{w}{u} + \frac{u}{w}$ and $\frac{u}{v} + \frac{v}{u}$ is

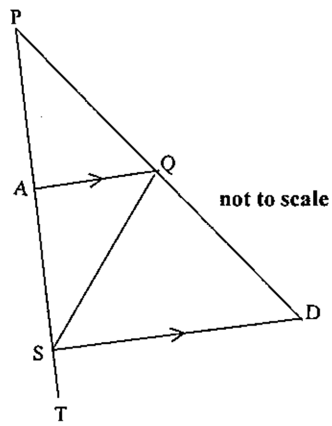
$B^2(x + 1)^3 + A^3(x + 1) + A^3 = 0$ **1**

(v) Evaluate $\frac{v}{w} + \frac{w}{v} + \frac{w}{u} + \frac{u}{w} + \frac{u}{v} + \frac{v}{u}$ **1**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



In the diagram PAST and PQD are straight lines and AQ || SD

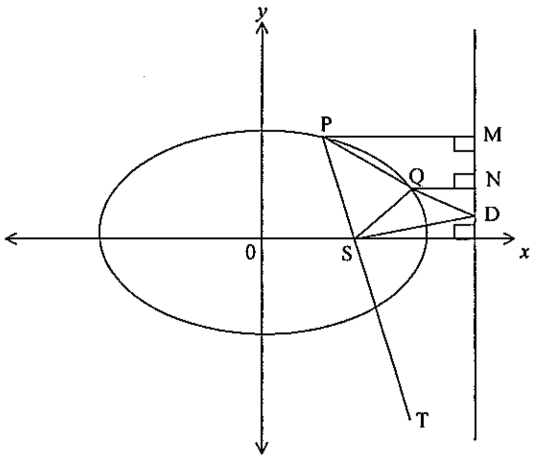
Further, $\dfrac{PS}{QS} = \dfrac{PD}{QD}$

(i) Explain why QS = AS **1**

(ii) Deduce that $\angle DSQ = \angle DST$ **2**

Question 6 continues next page

(b)



The diagram shows a chord PQ of an ellipse meeting a directrix at D. S is the corresponding focus. PM and QN meet this directrix at right angles at M and N, respectively.

(i) Show that $\dfrac{\text{PS}}{\text{QS}} = \dfrac{\text{PM}}{\text{QN}}$ **1**

(ii) Deduce that $\dfrac{\text{PS}}{\text{QS}} = \dfrac{\text{PD}}{\text{QD}}$ **1**

(iii) Deduce that $\angle\,\text{DSQ} = \angle\,\text{DST}$ **1**

(iv) Deduce that if the tangent at P meets the directrix at R then $\angle\,\text{PSR} = 90^\circ$ **2**

(c) (i) w is a complex root of $x^3 - 1 = 0$

(α) Explain why $\overline{w}$ is the other complex root and deduce that $1 + w + \overline{w} = 0$ **2**

(β) Show that $\overline{w} = w^2$ **1**

(ii) $A(x)$ and $B(x)$ are two polynomials with complex coefficients such that $A\left(x^3\right) + x\,B\left(x^3\right) \equiv \left(x^2 + x + 1\right)Q(x)$, where $Q(x)$ is a polynomial with complex coefficients.

(α) Prove that $A(1) = 0$ and $B(1) = 0$ **3**

(β) Deduce that $A\left(x^3\right) + x\,B\left(x^3\right)$ is divisible by $x^3 - 1$ **1**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) You are given the identity $\cos(A + B) + \cos(A - B) \equiv 2\cos A\cos B$

(i) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \cos 5x \cos 3x \quad dx$ **2**

(ii) Find the general solutions of the equation $\cos 5x + \cos 3x + 2\cos x = 0$ **3**

(b) A particle of unit mass moves on the x axis against a resistance numerically equal to $v^2 + v^3$, where v is its velocity. Initially the particle is travelling with velocity u, where $u > 0$.

(i) Prove that when the velocity is $\frac{u}{2}$ the distance X travelled by the particle is given by $X = \ln\left(\dfrac{2 + u}{1 + u}\right)$ **4**

(ii) Prove that if T is the time taken to travel the distance X then $u(T + X) = 1$ **4**

(iii) Thomas examined the motion of the particle more thoroughly. Thomas alleged that if the particle started at the origin then the velocity v, displacement x and time t were related by the equation

$$v = \frac{u}{ux + ut + 1}$$ **2**

By finding a suitable derivative, show that Thomas is correct.

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $u_n = \int_0^1 x^{2007}(1-x)^n \, dx, \qquad n = 0, 1, 2, \ldots$

(i) By considering $u_n - u_{n-1}$ show that $u_n < u_{n-1}$ **2**

(ii) Use integration by parts to show that $u_n = \dfrac{n}{2008 + n} u_{n-1}$, $n \geq 1$ **3**

(iii) Deduce that $u_n = \dfrac{2007!\, n!}{(2008 + n)!}$ **2**

(b) An extraordinary identity, due to the Swiss mathematician Leonard Euler (1707-83), states

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{for all real values of } \theta$$

(i) Show that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ **2**

(ii) Deduce that **3**

$$1 + 2\cos\theta + 2\cos 2\theta + \ldots + 2\cos n\theta = \frac{\sin\left(n + \frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta}, \quad n \geq 1$$

(iii) Hence or otherwise find $\displaystyle\lim_{\theta \to 0} \frac{\sin\left(n + \frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta}$ **1**

(iv) Use (ii) to show that **2**

$$1 + (2\cos\theta)^2 + (2\cos 2\theta)^2 + \ldots + (2\cos n\theta)^2 = 2n + \frac{\sin(2n + 1)\theta}{\sin\theta}, \quad n \geq 1$$

End of Examination

Standard Integrals

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$

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Mathematics Extension 2

Question (Marks)		Complex Numbers		Functions		Integration		Conics		Mechanics	
1	(15)						15				
2	(15)	(a), (b), (c)	9	(d)	6						
3	(15)					(b)	5	(a)	10		
4	(15)	(c)	5			(a), (b)	10				
5	(15)			(b)	8					(a)	7
6	(15)			(c)	7			(a), (b)	8		
7	(15)					(a)	5			(b)	10
8	(15)	(b)	8			(a)	7				
Total	(120)		22		21		42		18		17

TKS Extension 2 Trial 2007 — Solutions

Question 1

(a) $t = \tan\frac{x}{2}$ $x = 0,\ t = 0$

$x = \frac{\pi}{2},\ t = 1$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}(1+t^2)$

$$\therefore\ I = \int_0^1 \frac{2\,dt}{3(1+t^2)+1-t^2}$$

$$= \int_0^1 \frac{dt}{2+t^2} = \frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{t}{\sqrt{2}}\right]_0^1 = \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}$$

(b) (i) put $u = 1-x^2$; $x = 0,\ u = 1$

$\frac{du}{dx} = -2x$ $x = \frac{\sqrt{3}}{2},\ u = \frac{1}{4}$

$$\therefore\ I = -\frac{1}{2}\int_1^{\frac{1}{4}} \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\cdot 2\left[\sqrt{u}\right]_{\frac{1}{4}}^{1}$$

$$= \frac{1}{2}$$

(ii) $u = \cos^{-1}x$, $\frac{dv}{dx} = 1$

$\frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$, $v = x$

$$\therefore\ I = \left[x\cos^{-1}x\right]_0^{\frac{\sqrt{3}}{2}} + \int_0^{\frac{\sqrt{3}}{2}} \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \frac{\pi\sqrt{3}}{12} + \frac{1}{2}$$

(c) (i) Equating constants, $-2A - 2 = -12$

$\therefore A = 5$

(ii) $$I = \int_0^1 \frac{2x+5}{x^2+1} - \frac{2}{x-2} \, dx$$

$$= \int_0^1 \frac{2x}{x^2+1} + \frac{5}{x^2+1} - \frac{2}{x-2} \, dx$$

$$= \Big[\ln(x^2+1) + 5\tan^{-1}x - 2\ln|x-2|\Big]_0^1$$

$$= \ln 2 + \frac{5\pi}{4} - 0 - (0 + 0 - 2\ln 2)$$

$$= 3\ln 2 + \frac{5\pi}{4}$$

(d) $$I = \int \tan x \tan^2 x \, dx$$

$$= \int \tan x (\sec^2 x - 1) \, dx$$

$$= \int \tan x \cdot \sec^2 x - \frac{\sin x}{\cos x} \, dx$$

$$= \frac{\tan^2 x}{2} + \ln \cos x \quad (+c)$$

Question 2

(a) (i) $\left|\frac{z}{w}\right| = \frac{\sqrt{a^2+1}}{\sqrt{1+a^2}} = 1$

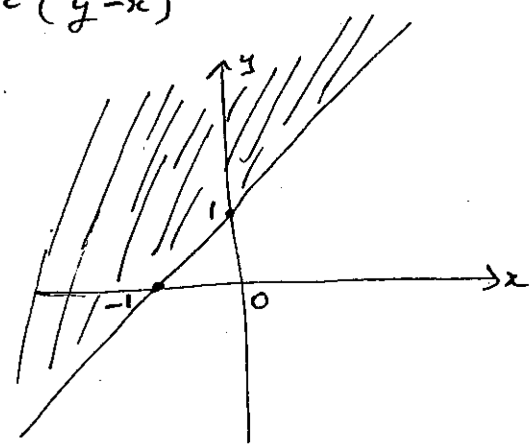
(ii) $zw = (a+i)(1+ai) = a - a + i(a^2+1)$

$= (a^2+1)i$

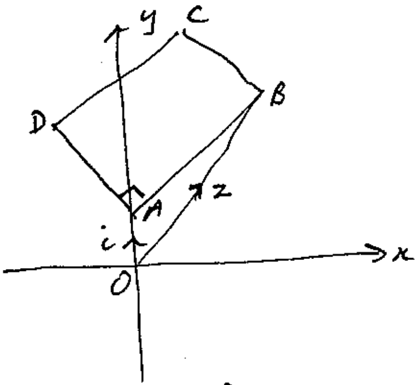
$\therefore \arg zw = \frac{\pi}{2}$

(b) $(1-i)(x+iy) = x + y + i(y-x)$

$\therefore y - x \geq 1$



(c) (i)



$\vec{AB} = z - i$

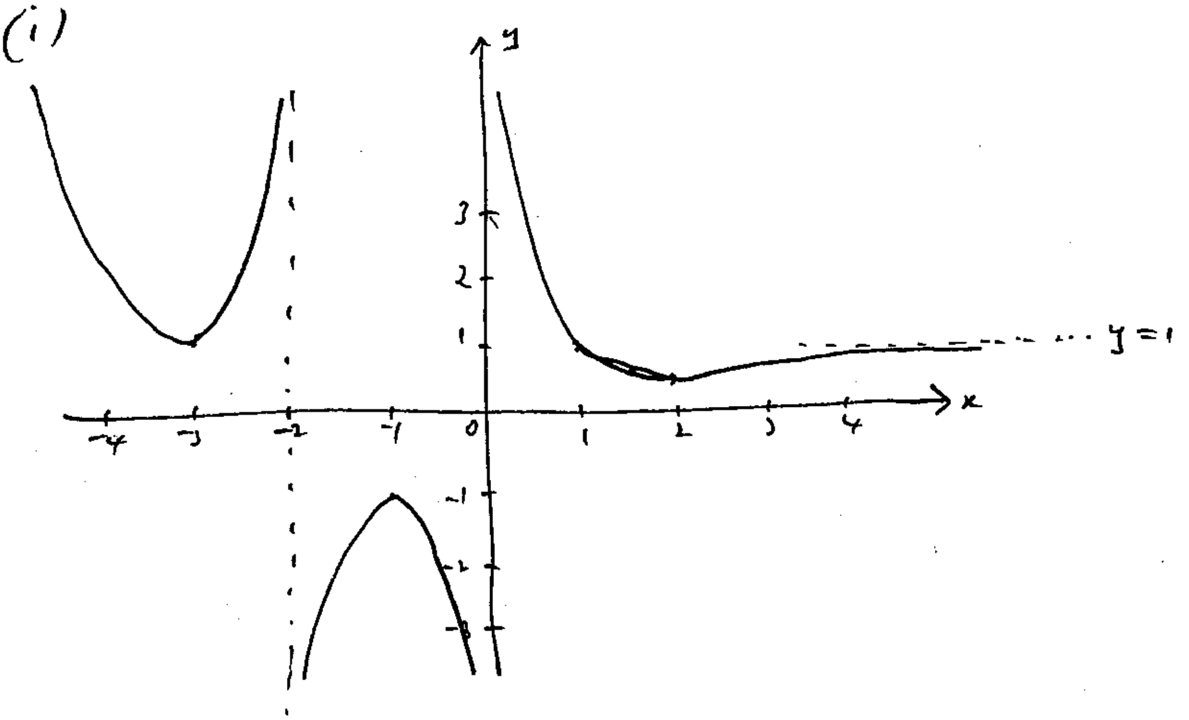
$\therefore \vec{AD} = i\vec{AB} = 1 + iz$

(ii) From (i), $\vec{BC} = 1 + iz$

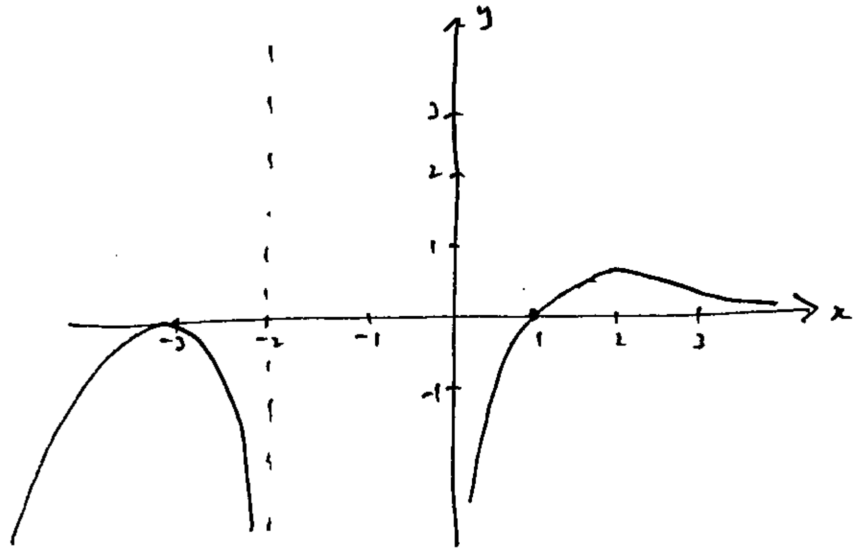
$\therefore \vec{OC} = \vec{OB} + \vec{BC} = z + 1 + iz$

$= 1 + (1+i)z$

(d) (i)



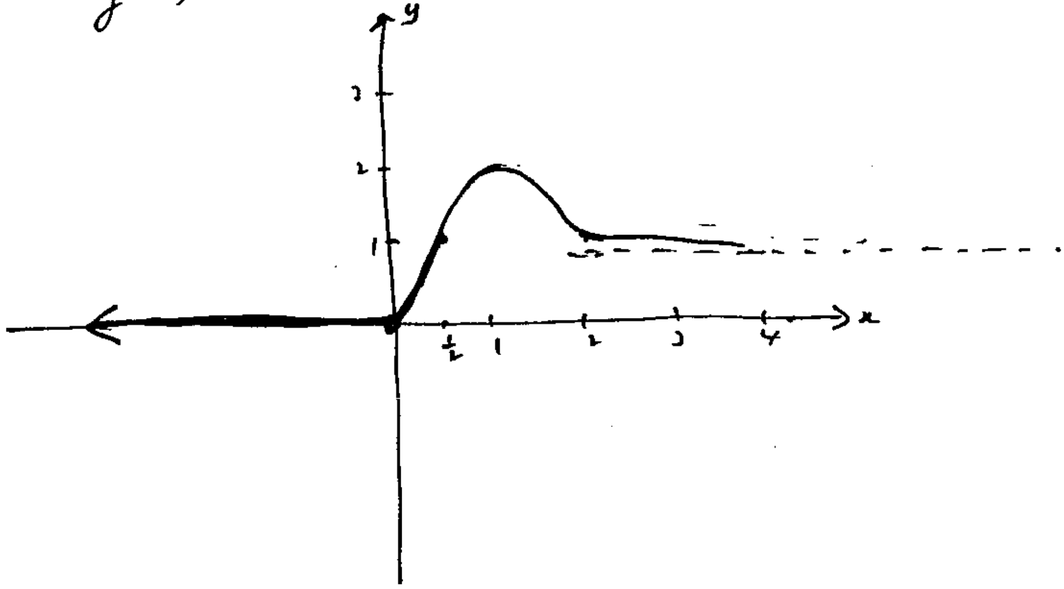
(ii)



(iii) If $x \geq 0$, $y = f(x + x) = f(2x)$

If $x \leq 0$, $y = f(x - x) = f(0) = 0$

$\therefore$

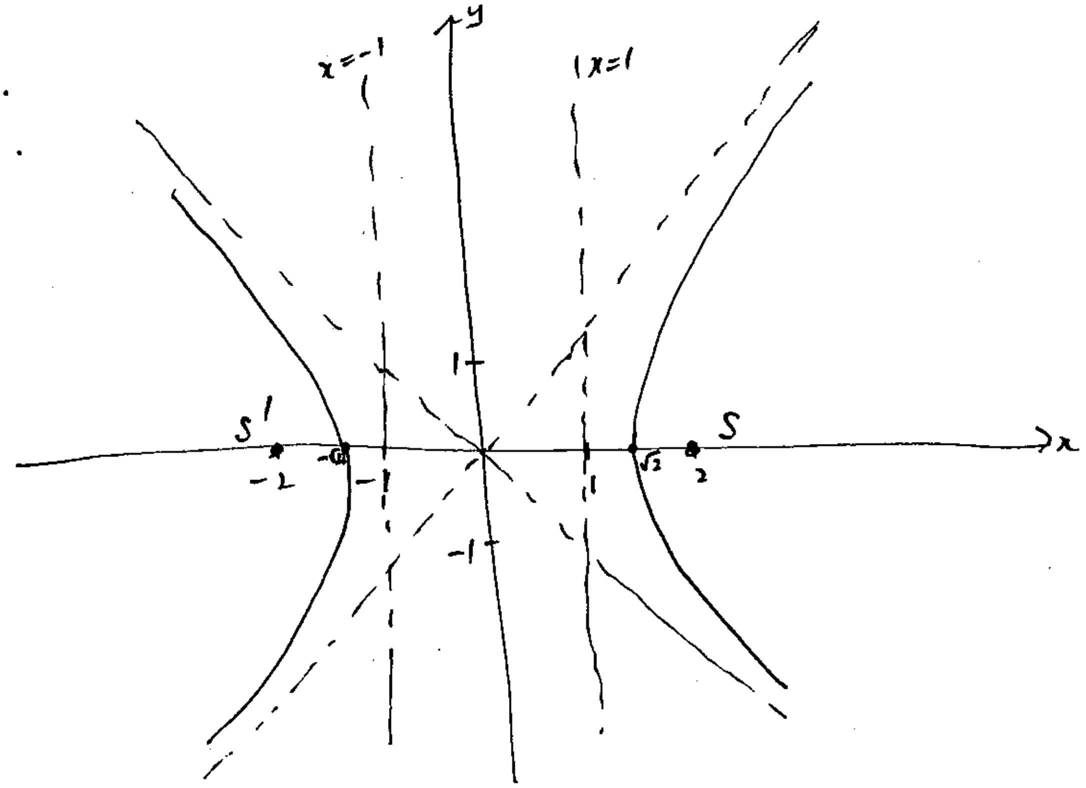


Question 3

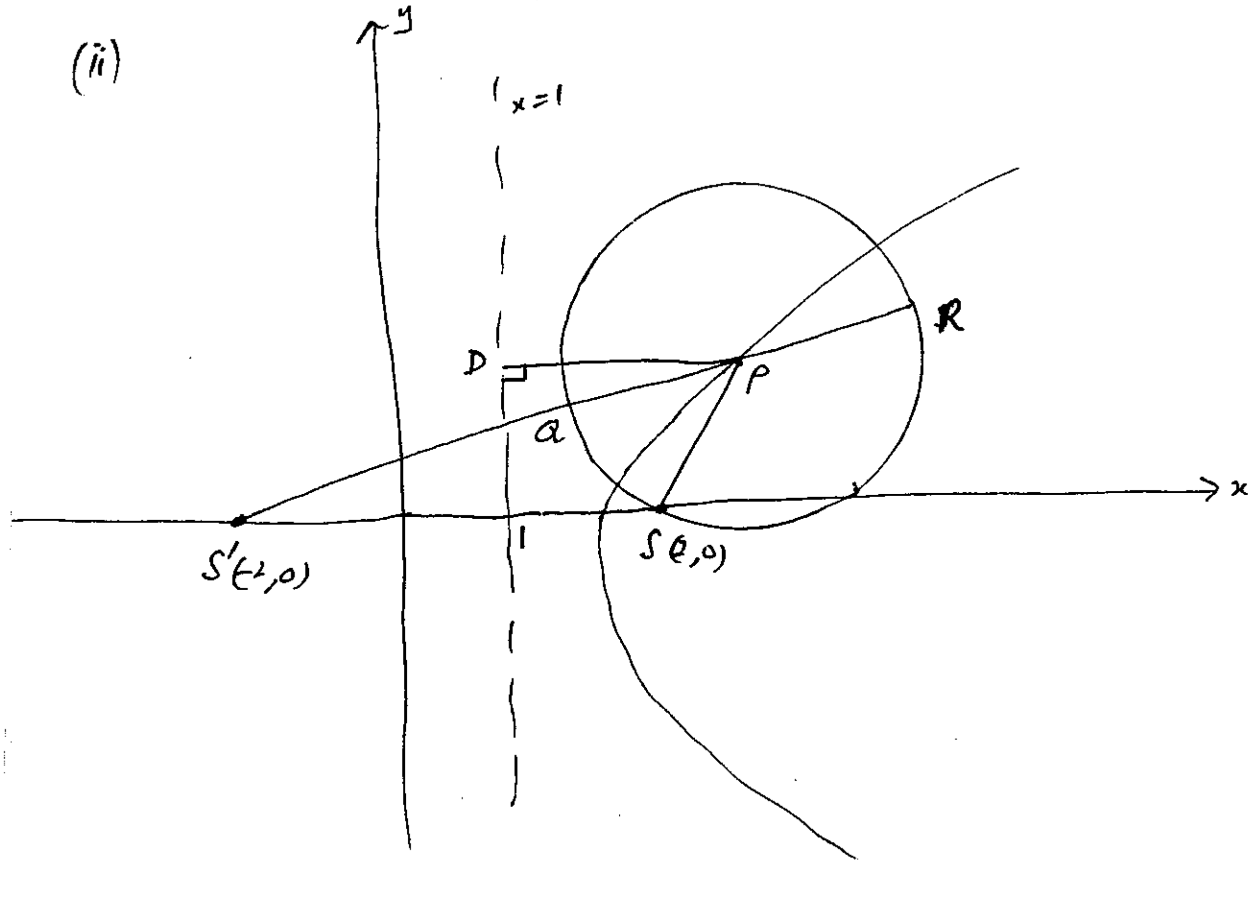
(a) (i) $c^2 = 2+2 = 4 \quad \therefore c = 2 \quad$ ie. $S = (2, 0)$

$e = \sqrt{2} \Rightarrow$ directrices are $x = \pm \frac{\sqrt{2}}{\sqrt{2}} = \pm 1$

$y = 0 \Rightarrow x = \pm \sqrt{2}$



(ii)



(α) $PS = e\,PD = \sqrt{2}\left(\sqrt{2}\sec\theta - 1\right) = 2\sec\theta - \sqrt{2}$

(β) gradient $QS = \dfrac{2\sqrt{2}\tan\theta}{(\sqrt{2}\sec\theta+1)\left[\dfrac{2}{\sqrt{2}\sec\theta+1} - 2\right]}$

$= \dfrac{\sqrt{2}\tan\theta}{1-(\sqrt{2}\sec\theta+1)} = -\dfrac{\tan\theta}{\sec\theta}$

Next, $2x - 2y\dfrac{dy}{dx} = 0$ on hyperbola

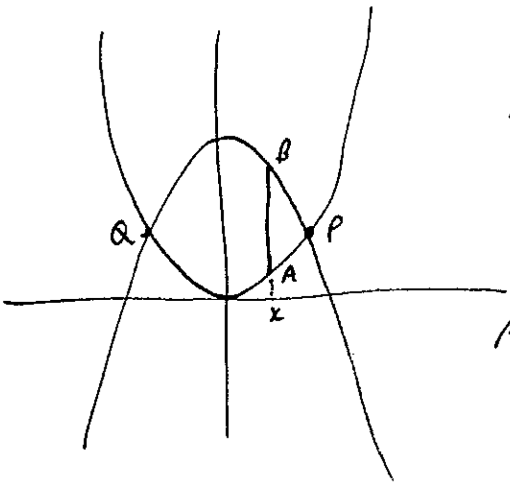
$\therefore \dfrac{dy}{dx} = \dfrac{x}{y} = \dfrac{\sec\theta}{\tan\theta}$ at P

$\therefore$ gradient normal $= -\dfrac{\tan\theta}{\sec\theta} \Rightarrow QS \parallel$ normal

(γ) Now, $\angle QSR = 90°$, angle in semi-circle

$\therefore RS \perp QS \Rightarrow RS \parallel$ tangent at P from (β)

(b)



at P, Q, $x^2 = 8 - x^2$
$x^2 = 4 \Rightarrow x = 2, -2$

$AB = 8 - x^2 - x^2 = 8 - 2x^2$

Area of semi-circle on AB as diameter
$= \frac{1}{2}\pi(4-x^2)^2$

$\therefore V = 2\displaystyle\int_0^2 \frac{1}{2}\pi(4-x^2)^2\,dx$

$= \pi\displaystyle\int_0^2 16 - 8x^2 + x^4\,dx$

$= \pi\left[16x - \dfrac{8x^3}{3} + \dfrac{x^5}{5}\right]_0^2 = \pi\left(32 - \dfrac{64}{3} + \dfrac{32}{5}\right) = \dfrac{256\pi}{15}$

Question 4

(a) (i) $\frac{d^2y}{dx^2} = \frac{1}{2}(4y^2+1)^{-\frac{1}{2}} \, 8y \, \frac{dy}{dx}$

$= 4y \, \frac{dx}{dy} \cdot \frac{dy}{dx} = 4y$

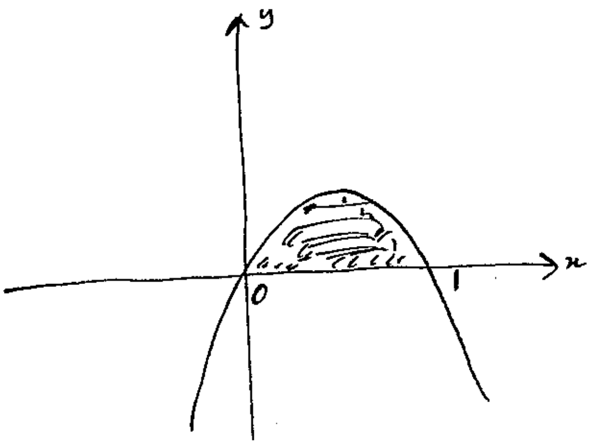
(ii) $\frac{dx}{dy} = \frac{1}{\sqrt{4y^2+1}} = \frac{1}{\sqrt{(2y)^2+1}}$

$\therefore x = \frac{1}{2} \ln\left(2y + \sqrt{4y^2+1}\right) + c$

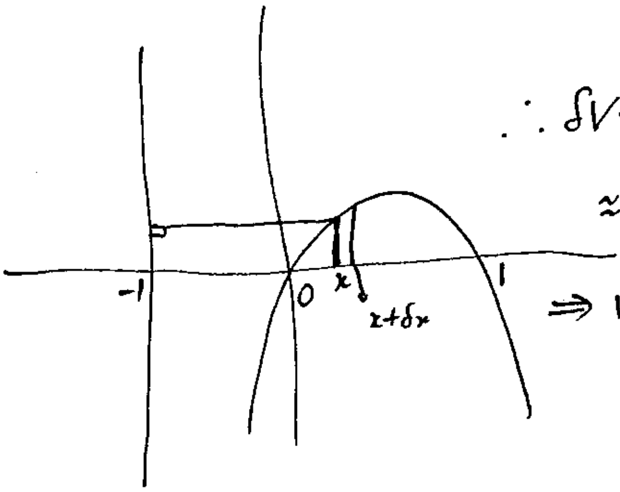
$: \; 0 = \frac{1}{2}\ln 1 + c, \quad c = 0$

$\therefore x = \frac{1}{2} \ln\left(2y + \sqrt{4y^2+1}\right)$

(b) (i)



(ii)



$\therefore \delta V \simeq \pi\left[(x+\delta x+1)^2 - (x+1)^2\right] y$

$\simeq \pi \left(2(x+1)\,\delta x\right) y$

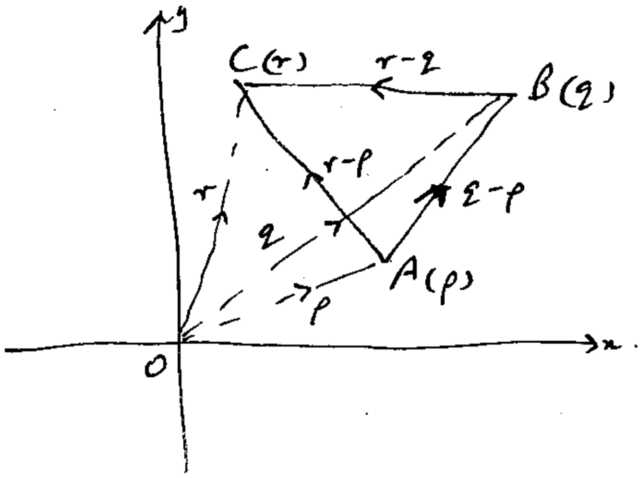
$\Rightarrow V = 2\pi \int_0^1 (x+1)(x-x^2)\,dx$

$= 2\pi \int_0^1 x - x^3 \, dx$

$= 2\pi \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{2}$

(c) (i) $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

(ii)



(α) $\vec{AC} = r - p,\ \vec{AB} = q - p$

$\therefore\ \vec{AC} = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\vec{AB}$

$\Rightarrow\ \angle CAB = 60^\circ$ & $|\vec{AC}| = |\vec{AB}|$

$\therefore\ \angle ACB = \angle ABC = 60^\circ$

$\therefore\ \Delta ABC$ is equilateral

$\therefore\ \vec{BA} = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\vec{BC}$

$\Rightarrow\ p - q = \frac{1}{2}(1 + i\sqrt{3})(r - q)$

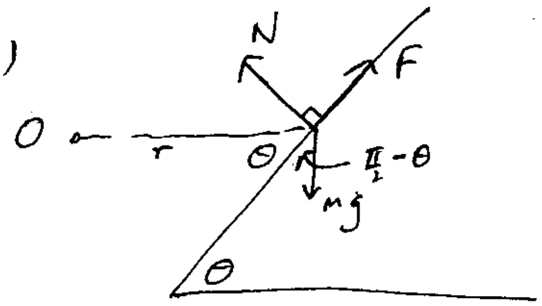
(β) $\therefore\ \dfrac{r-p}{p-q} = \dfrac{q-p}{r-q}$ from data & (α)

$\therefore\ r^2 - pr - qr + pq = -(p^2 - 2pq + q^2)$

ie $p^2 + q^2 + r^2 = pq + qr + rp$

Question 5

(a) (i)



Resolving in direction BA we have

$$\frac{mv^2}{r}\cos\theta = mg\cos\left(\frac{\pi}{2}-\theta\right) - F$$

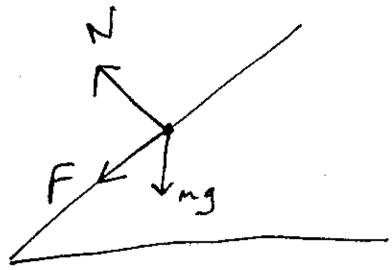
$$\Rightarrow F = mg\sin\theta - m\frac{v^2}{r}\cos\theta$$

(ii) Since $F > 0$, $g\sin\theta - \frac{v^2}{r}\cos\theta > 0$

i.e. $g\tan\theta - \frac{v^2}{r} > 0$

$\Rightarrow v^2 < rg\tan\theta$

(iii)



(iv) Resolving in the direction of N,

$$\frac{mv^2}{r}\cos\left(\frac{\pi}{2}-\theta\right) = N - mg\cos\theta$$

$$\Rightarrow N = mg\cos\theta + \frac{mv^2}{r}\sin\theta$$

(b) (i) $u^2+v^2+w^2 = (u+v+w)^2 - 2(uv+vw+wu)$

$$= 0^2 - 2A = -2A$$

(ii) $y = \dfrac{v^2+w^2}{vw}$ where $u^2+v^2+w^2 = -2A$ and $uvw = -B$

$$\therefore y = \frac{-2A-u^2}{-\frac{B}{u}} = \frac{2Au+u^3}{B}$$

or $u^3 + 2Au - By = 0$

(iii) Since u is a root then

$$u^3 + Au + B = 0$$

$\therefore$ from (ii), $Au - By - B = 0$

$$\therefore u = \frac{B(y+1)}{A}$$

(iv) From (iii) the equation is

$$\left(\frac{B}{A}(y+1)\right)^3 + A\frac{B}{A}(y+1) + B = 0$$

i.e. $B^2(y+1)^3 + A^3(y+1) + A^3 = 0$

or $B^2(x+1)^3 + A^3(x+1) + A^3 = 0$

(v) ie. the sum of the roots of (iv)

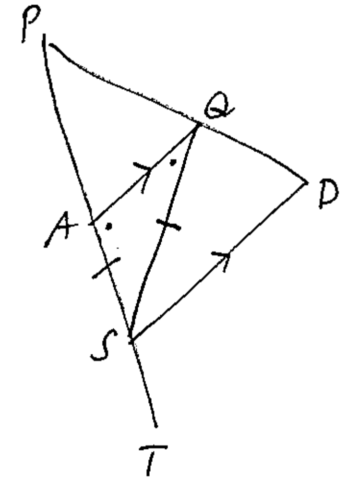
$$= -\frac{3B^2}{B^2} = -3$$

Question 6

(a) (i) $\dfrac{PD}{QD} = \dfrac{PS}{AS}$, ratio intercept theorem in // lines

$= \dfrac{PS}{QS}$, data $\quad \therefore \; QS = AS$

(ii)



From (i), $\angle SQA = \angle SAQ$, base angles isos Δ.

But $\angle SQA = \angle DSQ$, alt $\angle s$ in // lines

& $\angle SAQ = \angle DST$, corr. $\angle s$ in // lines

$\therefore \angle DSQ = \angle DST$

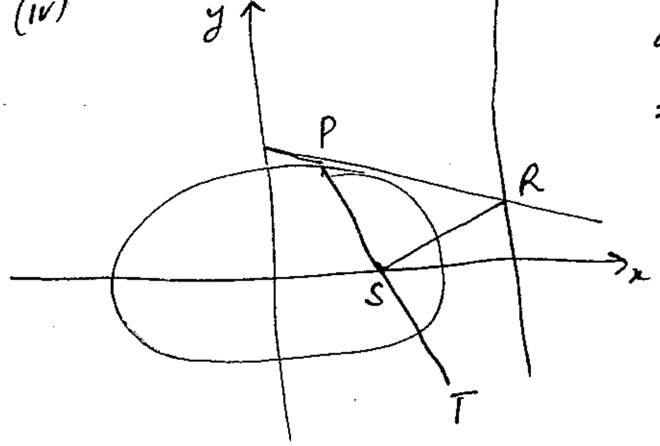
(b) (i) $\dfrac{PS}{QS} = \dfrac{ePM}{eQN} = \dfrac{PM}{QN}$, (focus-directrix defn of ellipse)

(ii) $\dfrac{PM}{QN} = \dfrac{PD}{QD}$ since $\Delta PMD \,|||\, \Delta QND$

$\therefore$ from (i), $\dfrac{PS}{QS} = \dfrac{PD}{QD}$

(iii) $\therefore \angle DSQ = \angle DST$ from (a), independent of A

(iv)



as $Q \to P$, $D \to R$

$\Rightarrow$ from (iii),

$\angle RSP = \angle RST$

But $\angle PST = 180°$

$\therefore \angle PSR = 90°$

(c) (i) (α) Since the coefficients of $x^3 - 1 = 0$ are real then the complex roots occur in conjugate pairs.

$\therefore \bar{\omega}$ is the other complex root

$\therefore 1 + \omega + \bar{\omega} = 0$ since sum of roots is 0

(β) $x^3 - 1 = (x-1)(x^2 + x + 1)$

$\Rightarrow 1 + \omega + \omega^2 = 0 \Rightarrow \bar{\omega} = \omega^2$, from ($\alpha$)

[Alternatives abound]

(ii) (α) $\therefore A(\omega^3) + \omega B(\omega^3) = 0$

$\therefore A(1) + \omega B(1) = 0$ since $1 + \omega + \omega^2 = 0$

Also, $A((\bar{\omega})^3) + \bar{\omega} B(\bar{\omega}^3) = 0$ since $1 + \bar{\omega} + \bar{\omega}^2 = 0$

ie $A(1) + \bar{\omega} B(1) = 0$

$\therefore (\omega - \bar{\omega}) B(1) = 0 \Rightarrow B(1) = 0$ since $\omega \neq \bar{\omega}$

$\therefore A(1) = 0$

(β) From (α), 1 is a root of $A(x^3) + x B(x^3) = 0$

$\Rightarrow A(x^3) + x B(x^3) \equiv (x^2 + x + 1)(x - 1) R(x)$

$= (x^3 - 1) R(x)$

ie: is divisible by $x^3 - 1$

Question 7

(a) (i) $\therefore I = \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos 8x + \cos 2x \, dx$

$$= \frac{1}{2}\left[\frac{\sin 8x}{8} + \frac{\sin 2x}{2}\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left(0 + \frac{1}{2} - 0\right) = \frac{1}{4}$$

(ii) For $\cos 5x + \cos 3x$ put $A + B = 5x$, $A - B = 3x$

$\Rightarrow A = 4x, \; B = x$

$\therefore$ we have $2\cos 4x \cos x + 2\cos x = 0$

i.e. $\cos x(\cos 4x + 1) = 0$

$\therefore \cos x = 0$ or $\cos 4x = -1$

$\therefore x = 2n\pi \pm \cos^{-1} 0$ or $4x = 2n\pi \pm \cos^{-1}(-1)$

i.e. $x = 2n\pi \pm \frac{\pi}{2}$ or $x = \frac{n\pi}{2} \pm \frac{\pi}{4}$, n an integer

(b) (i) $v\frac{dv}{dx} = -(v^2 + v^3)$

$\therefore \frac{dv}{dx} = -(v + v^2)$

$\Rightarrow \frac{dx}{dv} = -\frac{1}{v(1+v)} = \frac{1}{1+v} - \frac{1}{v}$

$\therefore X = \int_u^{\frac{u}{2}} \frac{1}{1+v} - \frac{1}{v} \, dv$

$$= \left[\ln(1+v) - \ln v\right]_u^{\frac{u}{2}}$$

$$= \left[\ln\left(\frac{1+v}{v}\right)\right]_u^{\frac{u}{2}} = \ln\left(\frac{1+\frac{u}{2}}{\frac{u}{2}}\right) - \ln\left(\frac{1+u}{u}\right)$$

$$= \ln\left(\frac{2+u}{u}\cdot\frac{u}{1+u}\right)$$

$$= \ln\left(\frac{2+u}{1+u}\right)$$

(ii) $\dfrac{dv}{dt} = -(v^2+v^3)$

$$\Rightarrow \frac{dt}{dv} = -\frac{1}{v^2(1+v)} \equiv \frac{Av+B}{v^2} + \frac{C}{1+v}$$

$$\Rightarrow (1+v)(Av+B) + Cv^2 \equiv -1$$

$$\therefore B=-1 \;,\; C=-1 \;,\; A=1$$

$$\therefore \quad T = \int_u^{\frac{u}{2}} \frac{v-1}{v^2} - \frac{1}{1+v}\, dv$$

$$= \int_u^{\frac{u}{2}} \frac{1}{v} - \frac{1}{1+v} - \frac{1}{v^2}\, dv$$

$$= -X - \int_u^{\frac{u}{2}} \frac{1}{v^2}\, dv \quad \text{from (i)}$$

$$= -X + \left[\frac{1}{v}\right]_u^{u/2}$$

ie. $T + X = \frac{2}{u} - \frac{1}{u} = \frac{1}{u}$

$$\therefore u(T+X) = 1$$

(iii) $v = u(ux+ut+1)^{-1}$

$$\therefore \frac{dv}{dt} = -u(ux+ut+1)^{-2}\left(u\frac{dx}{dt} + u\right)$$

$$= \frac{-u(uv+u)}{(ux+ut+1)^2}$$

$$\Rightarrow \frac{dv}{dt} = -\left(\frac{u}{ux+ut+1}\right)^2(v+1)$$

$$= -v^2(v+1)$$

$$= -(v^2+v^3)$$

= Thomas is correct

Question 8

(a) (i) $u_n - u_{n-1} = \int_0^1 x^{2007} \left[(1-x)^n - (1-x)^{n-1}\right] dx$

$$= \int_0^1 x^{2007}(1-x)^{n-1}\left[1-x-1\right] dx$$

$$= -\int_0^1 x^{2008}(1-x)^{n-1} dx$$

< 0 since $x^{2008}(1-x)^{n-1} \geq 0$ for $0 \leq x \leq 1$

i.e. $u_n < u_{n-1}$

(ii) Put $u = (1-x)^n$, $\frac{dv}{dx} = x^{2007}$

Then $\frac{du}{dx} = -n(1-x)^{n-1}$, $v = \frac{x^{2008}}{2008}$

$$\therefore u_n = \left[\frac{x^{2008}(1-x)^n}{2008}\right]_0^1 + \frac{n}{2008}\int_0^1 x^{2008}(1-x)^{n-1} dx$$

$$= 0 + \frac{n}{2008}(u_{n-1} - u_n) \quad \text{from (i)}$$

$$\therefore u_n\left(1 + \frac{n}{2008}\right) = \frac{n}{2008} u_{n-1}$$

i.e. $u_n = \frac{n}{2008+n} u_{n-1}$

(iii) From (ii), $u_n = \frac{n}{2008+n} \cdot \frac{n-1}{2007+n} \cdot \frac{n-2}{2006+n} \cdots \cdot \frac{1}{2009} u_0$

where $u_0 = \int_0^1 x^{2007} dx = \frac{1}{2008}$

$$\therefore u_n = \frac{n}{2008+n} \cdot \frac{n-1}{2007+n} \cdots \frac{2}{2010} \cdot \frac{1}{2009} \cdot \frac{1}{2008} \cdot \frac{2007 \cdot 2006 \cdots 2 \cdot 1}{2007 \cdot 2006 \cdots 2 \cdot 1}$$

$$= \frac{n!\,2007!}{(2008+n)!}$$

(b) (i) $e^{i\theta} = \cos\theta + i\sin\theta$

$\therefore\ e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

$\therefore\ 2\cos\theta = e^{i\theta} + e^{-i\theta}$ and $2i\sin\theta = e^{i\theta} - e^{-i\theta}$

$\therefore\ \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

(ii) from (i),

$LS = 1 + (e^{i\theta} + e^{-i\theta}) + (e^{2i\theta} + e^{-2i\theta}) + \dots + (e^{ni\theta} + e^{-ni\theta})$

$= e^{-in\theta} + e^{-i(n-1)\theta} + \dots + e^{-i\theta} + 1 + e^{i\theta} + \dots + e^{in\theta}$

$= \dfrac{e^{-in\theta}\left((e^{i\theta})^{2n+1} - 1\right)}{e^{i\theta} - 1}$ using the geometric series

$= \dfrac{e^{i(n+1)\theta} - e^{-in\theta}}{e^{i\theta} - 1}$

$= \dfrac{e^{i(n+\frac{1}{2})\theta} - e^{-i(n+\frac{1}{2})\theta}}{e^{\frac{i\theta}{2}} - e^{-i\theta/2}}$

$= \dfrac{2i\sin(n+\frac{1}{2})\theta}{2i\sin\frac{1}{2}\theta}$ from (i)

$= \dfrac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}$

(iii) Limit $= 1 + 2 + 2 + \cdots + 2$, n 2's

$= 2n + 1$

(iv) LS $= 1 + 2(2\cos^2\theta) + 2(2\cos^2 2\theta) + \cdots + 2(2\cos^2 n\theta)$

$= 1 + 2(1+\cos 2\theta) + 2(1+\cos 4\theta) + \cdots + 2(1+\cos 2n\theta)$

$= 2n + (1 + 2\cos 2\theta + 2\cos 4\theta + \cdots + 2\cos 2n\theta)$

$= 2n + \dfrac{\sin(n+\frac{1}{2})2\theta}{\sin(\frac{1}{2}\cdot 2\theta)}$ from (ii)

$= 2n + \dfrac{\sin(2n+1)\theta}{\sin\theta}$